Pin structures on manifolds quotiented by discrete groups

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Abstract. *We present a method of classifyng and constructing bundle prolongations on manifolds of the form M/Γ, where Γ is a discrete group, in terms of data on M, and apply it to two-dimensional closed surfaces. We describe all eight inequivalent double coverings of the pseudo-orthogonal group and use these for constructing pin structures on (products of) non-orientable manifolds.*

1. INTRODUCTION AND NOTATION

In this paper we shall describe spinor fields on quotient manifolds of the type M/Γ, where Γ is a discrete group of transformations of M (acting properly and without fixed points). To do this, we first study spin-structures on M/Γ and their relation to spin-structures on M. Existence conditions and the number of inequivalent structures are characterized by known topological conditions and in this paper we shall reinterpret these conditions in a group-theoretical language and give a method of classifying and constructing all the spin structures on M/Γ in terms of analogous objects and data on M (section 2).

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Our method however allows us to treat not only spin structures, corresponding to the exact sequence

$$1 \to \mathbf{Z}_2 \to Spin(p, q) \to SO(p, q) \to 1,$$

but more generally, the case of an arbitrary discrete central extension of a Lie group G

$$1 \to K \to \widetilde{G} \xrightarrow{\rho} G \to 1$$

and its associated «$\widetilde{G}$-structure». This includes *pin* (cf. section 3 and 4) and $spin_c$ structures as well as couplings of spinors to non-abelian gauge fields. More precisely a $\widetilde{G}$-structure (or prolongation of some principal G-bundle F to the structure group $\widetilde{G}$) on M/Γ – denoted by $\eta : \widetilde{F} \to F$ or $(\widetilde{F}, \eta)$ – consists of a principal $\widetilde{G}$-bundle $\widetilde{\pi} : \widetilde{F} \to M/\Gamma$, a principal G-bundle $\pi : F \to M/\Gamma$ and a strong bundle morphism $\eta : \widetilde{F} \to F$ with the properties $\pi \circ \eta = \widetilde{\pi}$ and $\eta(\widetilde{e}\widetilde{h}) = \eta(\widetilde{e})\, \rho(\widetilde{h})$, where $\widetilde{e}\widetilde{h}$ (resp. $\widetilde{\eta}(\widetilde{e})\, \rho(\widetilde{h})$) denotes the principal right action of $\widetilde{G}$ on $\widetilde{F}$ (resp. of G on F).

If we define two $\widetilde{G}$-structures $(\widetilde{F}, \eta)$ and $(\widetilde{F}', \eta')$ to be equivalent iff there exists a strong bundle isomorphism $\beta : \widetilde{F} \to \widetilde{F}'$ which interwines η and η', $\eta' = \eta \circ \beta$, then the inequivalent $\widetilde{G}$-structures are labelled by the first cohomology group $H^1(M/\Gamma, K)$ of M/Γ with coefficients in K [1].

In this setting fields are sections of bundles associated with some representation D of $\widetilde{G}$ in a linear space V or, what is actually the same, equivariant functions from $\widetilde{F}$ to V

$$\psi(\widetilde{e}\widetilde{h}) = D(h^{-1})\, \psi(\widetilde{e}).$$

This includes fields associated to F, since any representation of G can be thought of as a representation of $\widetilde{G}$ (via composition with ρ) and any G-equivariant function on F as a $\widetilde{G}$-equivariant function on $\widetilde{F}$ (by composition with η) but not vice versa. For spinors, F is the bundle of oriented orthonormal frames, and integer spin fields can be identified with ordinary tensors on M/Γ. Furthermore, fields on M/Γ correspond – in a way which will be made precise below – to Γ-invariant fields on M. Additionally any connection ω on F gives rise in a natural way to a connection $\widetilde{\omega}$ (and a covariant derivative) on $\widetilde{F}$ through

$$\widetilde{\omega} := (T\rho^{-1})\, \eta^* \omega$$

where $T\rho$ (the derivative of ρ) is the isormorphism ($Ker\ \rho$ is discrete!) of Lie algebras of $\widetilde{G}$ and G, and η^* is the pull-back of forms.

At the end of section 2 we comment on the possibility of extending our classification and construction scheme to orbifolds of the form M/Γ, where Γ may have fixed points, and illustrate the difficulties by some examples.

In section 3 we study – in order to properly handle reflections and «spinors» on non-orientable manifolds – all the double coverings of the pseudo-orthogonal group $0(p, q)$. There are eight of them if we identify the extensions which not only are isomorphic as groups but also the isomorphism interwines the covering homomorphisms, i.e. the following diagram commutes

$$\begin{array}{ccc} \widetilde{G} & \longrightarrow & \widetilde{G}' \\ & \searrow \quad \swarrow & \\ & G & \end{array}$$

It turns out that this is the right notion of equivalence of two central extensions; for instance $\widetilde{G}$ or $\widetilde{G}'$-structures may have distinct existence conditions even if $\widetilde{G}$ and $\widetilde{G}'$ are isomorphic as groups but not equivalent in the above sense. The reader who accepts the formula (3.3) may skip the boring details which are included since it is often claimed that $G = 0(p, q), p, q > 1$, has only four $\mathbf{Z}_2$-extensions and since the six extensions which do not arise from Clifford algebras seem to have attracted little attention so far (see however [2]).

The results obtained up to then will be illustrated by some examples (including Riemann surfaces and pin structures on non-orientable manifolds) in section 4. Section 5 summarizes our results and addresses some open problems.

2. THE RELATION BETWEEN $\widetilde{G}$-STRUCTURES ON M AND M/Γ

Assume that we have a $\widetilde{G}$-structure $\eta : \widetilde{F} \to F$ on M/Γ. We can use the canonical projection $p : M \to M/\Gamma$ to pull back these bundles to M. Thus we obtain

$$\widetilde{F}_M := p^*\widetilde{F} = ((x, \tilde{e}) \in M \times \widetilde{F} : p(x) = \tilde{\pi}(\tilde{e}))$$

$$F_M := p^*F = ((x, e) \in M \times F : p(x) = \pi(e)$$

with the obvious projections $\pi_M : (x, e) \to x$ and $\tilde{\pi}_M : (x, \tilde{e}) \to x$. Then

$$\eta_M : \widetilde{F}_M \to F_M, \quad (x, \tilde{e}) \to (x, \eta(\tilde{e}))$$

defines a $\widetilde{G}$-structure on M ($\widetilde{F}_M$ and F_M are principal $\widetilde{G}$- and G-bundles on M and η_M has the required properties). Since it originates from a $\widetilde{G}$-structure on M/Γ, $\eta_M : \widetilde{F}_M \to F_M$ is Γ-invariant in the following sense:

i) There exists a subgroup $\{ u_\gamma \in \text{Aut } F_M : \pi_M \circ u_\gamma = \gamma \circ \pi_M, \gamma \in \Gamma \}$ of Aut F_M which is isomorphic to Γ (we identify these two groups and call u_γ a lift of γ). In the case at hand u_γ is simply given by

$$u_\gamma : (x, e) \to (\gamma(x), e)$$

ii) The automorphism group of $\widetilde{F}_M$ has a subgroup $\widetilde{\Gamma} = \{ \tilde{u}_\gamma \in \text{Aut } \widetilde{F}_M : \eta_M \circ \tilde{u}_\gamma = u_\gamma \circ \eta_M \}$ which covers Γ. The homomorphism $\kappa : \tilde{u}_\gamma \to u_\gamma$ makes the following sequence of groups exact

$$(2.1) \qquad 1 \to K \to \widetilde{\Gamma} \overset{\kappa}{\to} \Gamma \to 1,$$

where $K \subset \widetilde{G}$ is identified with the right multiplication by elements of K (this is a vertical automorphism, since K is central by assumption).

iii) There is a group homomorphism $\sigma : \Gamma \to \widetilde{\Gamma}, \gamma \to \sigma(\gamma)$, acting as

$$\sigma(\gamma) : (x, \tilde{e}) \to (\gamma(x), \tilde{e}),$$

with the property $\kappa \circ \sigma = id_\Gamma$, i.e. the exact sequence (2.1) splits.

Our main result is that this reasoning can be inverted and $\widetilde{G}$-structures on M/Γ can be put in 1 : 1 correspondence with $\widetilde{G}$-structures on M satisfying (i-iii) and splittings of (2.1).

THEOREM: Any $\widetilde{G}$-structure on M/Γ can be obtained as the quotient of a Γ-invariant $\widetilde{G}$-structure on M. There is a bijective correspondence between inequivalent $\widetilde{G}$-structures on M/Γ and $\{\Gamma$-invariant $\widetilde{G}$-structures on $M\} \times \mathrm{Hom}(\Gamma, K)$, i.e.: the quotients $\eta_\sigma : \widetilde{F}_M/\sigma(\Gamma) \to F_M/\Gamma$ and $\eta'_{\sigma'} : \widetilde{F}'_M \to F_M/\Gamma$ are equivalent if, and only if, $\eta : \widetilde{F}_M \to F_M$ and $\eta' : \widetilde{F}'_M \to F_M$ are and, moreover, $\sigma = \sigma'$.

REMARKS.

A) If F_M is the (orthonormal or linear) frame bundle of M, u_γ is given by the derivative (or tangent map) of γ. Another case, in which the lift of $\gamma \in \Gamma$ will always exist, is provided by a trivial principal bundle. Little however is known about the general case.

B) If M is simply connected, $\pi_1(M/\Gamma) \cong \Gamma$ and $\widetilde{G}$-structures on M/Γ are known to be classified by $H^1(M/\Gamma, K)$. In our framework we recover this result as follows. Since $\pi_1(M) = 0$, M has a unique $\widetilde{G}$-structure, which is therefore automatically left invariant. If it splits, $\widetilde{G}$-structures on M/Γ are labelled by elements of

$$\mathrm{Hom}(\Gamma, K) = \mathrm{Hom}(\pi_1(M/\Gamma), K) = \mathrm{Hom}(H_1(M/\Gamma), K) = H^1(M/\Gamma, K)$$

Proof of the theorem: Let $\eta_M : \widetilde{F}_M \to F_M$ be a $\widetilde{G}$-structure on M which is Γ-invariant in the sense above, and let σ_0 be a splitting of the sequence 2.1. It defines a bijective correspondence between $\mathrm{Hom}(\Gamma, K)$ and the set of all splittings by $h \to \sigma_h(\gamma)$ with

$$\sigma_h(\gamma) : \tilde{e} \to \sigma_0(\gamma)\ \tilde{e}h(\gamma) \quad \text{for } h \in \mathrm{Hom}(\Gamma, K).$$

For any splitting σ, $\widetilde{F}_M/\sigma(\Gamma)$ and F_M/Γ are principal $\widetilde{G}$- and G-bundles over M/Γ with the projections

$$\tilde{\pi} : [\tilde{e}_x]_{\sigma(\Gamma)} \to \pi(x) \quad \text{and} \quad \pi : [e_x]_\Gamma \to \pi(x),$$

where $\tilde{e}_x \in \tilde{\pi}_M^{-1}(x)$, $e_x \in \pi_M^{-1}(x)$, and [ ] denotes the equivalence class in the quotient bundles. The quotient bundle homomorphism

$$\eta_\sigma : \tilde{F}_M/\sigma(\Gamma) \to F_M/\Gamma, \ [\tilde{e}_x]_{\sigma(\Gamma)} \to [\eta(\tilde{e}_x)]_\Gamma$$

then defines a $\tilde{G}$-structure on M/Γ.

We shall now show that the mapping Q : {Γ-invariant $\tilde{G}$-structures on M} $\times$ Hom(Γ, K) $\to$ {$\tilde{G}$-structures on M/Γ} is injective. To this end take two $\tilde{G}$-structures $\eta_M : \tilde{F}_M \to F_M$ and $\eta'_M : \tilde{F}'_M \to F_M$ and two splittings σ and σ'. Assume that $\eta_\sigma : \tilde{F}_M/\sigma(\Gamma) \to F_M/\Gamma$ and $\eta_{\sigma'} : \tilde{F}'_M/\sigma'(\Gamma) \to F_M/\Gamma$ are equivalent. Then it can be seen that $(\tilde{F}_M, \eta_M)$ and $(\tilde{F}'_M, \eta'_M)$ are themselves equivalent.

Furthermore – as we shall now proceed to show – this implies $\sigma = \sigma'$. In order to see that, assume that there is a strong $\tilde{G}$-equivariant bundle isomorphism $\beta : \tilde{F}_M/\sigma(\Gamma) \to \tilde{F}'_M/\sigma'(\Gamma)$ inter-twining η_σ and $\eta_{\sigma'}$ i.e. $\eta_\sigma = \eta_{\sigma'} \circ \beta$, and that there is at least one $\gamma \in \Gamma$ with $\sigma(\gamma) \neq \sigma'(\gamma)$. Consider a path $L : [0, 1] \to M$ with $L(0) = x$ and $L(1) = \gamma(x)$ for some $x \in M$. Then above this path there are $\tilde{\beta}_s \in \tilde{G}$, continuous in s for $s \in [0, 1]$, such that

$$\beta([\tilde{e}_s]_{\sigma(\Gamma)}) = ([\tilde{e}_s \tilde{\beta}_s]_{\sigma'(\Gamma)}) \quad \text{for} \quad e_s \in \pi^{-1}(L(s)).$$

The condition that β is well defined (i.e. independent of the representative of the equivalence class) may be expressed as

$$\sigma'(\gamma)\, \tilde{e}_0\, \hat{\beta}_0 = \sigma(\gamma)\, \tilde{e}_0\, \hat{\beta}_1 .$$

Next, from $\eta_\sigma = \eta_{\sigma'} \circ \beta$ it follows that $\rho(\hat{\beta}_s) = 1$ and therefore $\hat{\beta}_s \in K$. Since $\hat{\beta}_s$ is continuous and K discrete, $\hat{\beta}_0 = \hat{\beta}_1$ and thus

$$\sigma'(\gamma)\, \tilde{e}_0 = \sigma(\gamma)\, \tilde{e}_0 .$$

Since $\tilde{e}_0$ was arbitrary, this implies $\sigma = \sigma'$, in contradiction with the assumption. Therefore Q is injective.

It only remains to prove that Q is surjective, i.e. that each $\tilde{G}$-structure on M/Γ can be obtained as a quotient of a $\tilde{G}$-structure on M which is Γ-invariant. Given a $\tilde{G}$-structure $\eta : \tilde{F} \to F$ on M/Γ, pull it back to M to obtain the $\tilde{G}$-structure $\eta_M : \tilde{F}_M \to F_M$ defined in section 1. Quotient $\tilde{F}_M$ by Γ and F_M by $\sigma(\Gamma)$, where $\sigma(\gamma) : (x, \tilde{e}) \to (\gamma(x), \tilde{e})$ (c.f. the property iii) and define the quotient bundle morphism

$$\eta_\sigma : \tilde{F}_M/\sigma(\Gamma) \to F_M/\Gamma, [(x, \tilde{e})]_{\sigma(\Gamma)} \to [(x, \eta(\tilde{e})]_\Gamma . \tag{2.2}$$

The bundle isomorphism $\tilde{F}_M/\sigma(\Gamma) \to \tilde{F}$ by $[(x, \tilde{e})]_{\sigma(\Gamma)} \to \tilde{e}$ shows that (2.2) and $\eta : \tilde{F} \to F$ are equivalent (we identify F_M/Γ with F by $[(x, e)]_\Gamma \to e$. Thus Q is surjective and this finishes the proof of the theorem. ∎

REMARKS

A) The isomorphism $H_1(M, K) = \mathrm{Hom}(\pi_1(M), K)$ shows that it suffices to check the consistency of the constructed bundles on loops (this method will also be used in the examples of section 4).

B) Describing fields in terms of equivariant functions (section 1) we schematically have the following situation:

$$\begin{array}{ccc} \widetilde{F}_M & \overset{\widetilde{q}}{\rightarrow} \widetilde{F} & \overset{\psi}{\rightarrow} V \\ \eta_M \downarrow & \downarrow \eta & \\ F_M & \rightarrow F & \\ \pi_M \downarrow & \downarrow \pi & \\ M & \rightarrow M/\Gamma & \end{array}$$

Given a local section $\widetilde{e}$ of $\widetilde{F}$, we can pull back ψ to M/Γ. Furthermore via $\widetilde{q}$ we can regard ψ as an equivariant function $\widetilde{\psi} : \widetilde{F}_M \rightarrow V$, $\widetilde{\psi}(\widetilde{e}_M) = \psi([\widetilde{e}_M]_\sigma)$. Let us now investigate its behaviour under Γ: choose $\widetilde{u}_\gamma \in \widetilde{\Gamma}$ covering a lift u_γ of $\gamma \in \Gamma$. Then choose frames $\widetilde{e}_x$ and $\widetilde{e}_{\gamma(x)} \in F_M$ above x and γx, respectively; $\widetilde{u}_\gamma$ relates these frames up to a translation $\hat{u}(x) \in \widetilde{G}$ in the fibres

$$\widetilde{u}_\gamma \widetilde{e}_x = \widetilde{e}_{\gamma(x)}\, \hat{u}(x)$$

and

$$\widetilde{\psi}(\widetilde{u}_\gamma \widetilde{e}_x) = \widetilde{\psi}(\widetilde{e}_{\gamma(x)}\, \hat{u}(x)) = D^{-1}(\hat{u}(x))\, \widetilde{\psi}(\widetilde{e}_{\gamma(x)}).$$

Thus

$$\widetilde{\psi}(\widetilde{e}_{\gamma(x)}) = D(\hat{u}(x))\, \widetilde{\psi}(\widetilde{e}_x) \tag{2.3}$$

and if we choose the frames $\widetilde{e}_{\gamma(x)}$ in such a way that $\hat{u}(x) = 1$, we see in which sense fields on M/Γ correspond to invariant fields on M. If we choose a different splitting σ' instead of σ, related to σ by an element h of $\mathrm{Hom}(\Gamma, K)$, (i.e. a different G-structure) (2.3) is replaced by

$$\widetilde{\psi}(\widetilde{e}_{\gamma(x)}) = D(h(\gamma)\, \widetilde{u}\ \ (x))\, \widetilde{\psi}(\widetilde{e}_x)$$

giving rise to twisted periodicity conditions along those loops in M/Γ which lift to paths from x to $\gamma(x)$ in M.

C) It is tempting to ask whether the procedure and results outlined in this section can be generalized to the case, where the action of Γ on M has fixed points. It is well known that in this case the resulting space M/Γ belongs to a class of orbifolds [3], or V-manifolds [4] which are globally of the form M/Γ, where Γ is a discrete group acting properly discontinuously but perhaps with fixed points. Apart from the interest in its own right, this question may be of relevance for fields arising from a superstring which propagates in an orbifold (c.f. [5]).

We make some scattered remarks illustrating the limitations of our method:

a) If the theorem were true for orbifolds without reservations, no spin-structures could exist on spaces of the form $\mathbb{C}P_2/\Gamma$, since they could be pulled back to yield spin-structures on $\mathbb{C}P_2$ – a contradiction. We have however the remarkable result that [6] $\mathbb{C}P_2/\mathbf{Z}_2 = S^4$, where $\mathbf{Z}_2$ acts as a complex conjugation (in homogeneous coordinates on $\mathbb{C}P_2$) and the fixed point set of $\mathbf{Z}_2$ is $\mathbf{R}P_2 \subset \mathbb{C}P_2$. However it is well known that S^4 admits a spin structure, namely

$$Spin(5) \to SO(5) \to S^4 = SO(5)/SO(4).$$

b) Although it is true that quotients of principal bundles over M are «principal V-bundles» [4] over M/Γ, we cannot conclude that all principal V-bundles over M/Γ arise in this way, since pullbacks of these will as a rule not give rise to ordinary principal bundles on M.

c) Even very simple orbifolds like $\mathbb{R}^3/\mathbf{Z}_2$ ($\mathbf{Z}_2$ being a point-reflection) may fail to be topological manifolds, however, many theorems developed in the smooth-manifold context can be generalized to orbifolds. Whence a classification of G-structures on orbifolds may –despite naive appearance – nevertheless be possible along the lines of the beginning of this section.

3. THE DOUBLE COVERINGS OF 0(p, q)

In this section we shall describe the double coverings of the pseudo-orthogonal groups $0(p, q)$ (preserving diag $(++\ldots+--\ldots-)$), which are equal to the standard double covering

$$\rho : Spin_0(p, q) \to SO_0(p, q), \tag{3.1}$$

defined in terms of Clifford algebras over the connected component $SO_0(p, q)$ of $0(p, q)$. Since $0(p, q) \cong 0(q, p)$ we can restrict our attention to the case $p \geqslant q$.

Consider first the case $q = 0$; $0(n, 0)$ will be denoted by $0(n)$ and has two connected components: one containing the identity and one the reflection of – say – the first axis. Since the covering of the identity component is fixed by (3.1) and the square of the element e_1 covering the reflection has to cover the identity of $SO_0(p, q)$, there are exactly two inequivalent coverings of $0(n)$. We denote them

$$\rho^{\pm} : Pin^{\pm}(n) \to 0(n)$$

according to $(e_1)^2 = \pm 1$. They can both be described in terms of Clifford algebras, or, equivalently by

$$Pin^{\pm}(n) = (Spin(n) \times 4^{\pm} / \mathbf{Z}_2, \tag{3.1}$$

$$Pin^{\pm}(n) \ni [a, h] \overset{\rho^{\pm}}{\to} (\rho(a), \nu^{\pm}(h)) \in SO(n) \odot \mathbf{Z}_2 \cong O(n),$$

where

$$4^{+} = \mathbf{Z}_2 \times \mathbf{Z}_2 \quad (h, h') \overset{\nu^{+}}{\to} hh' \in \mathbf{Z}_2,$$

$$4^{-} = \mathbf{Z}_4 \quad h \overset{\nu^{-}}{\to} h^2 \in \mathbf{Z}_2,$$

and $\odot$ denotes the semidirect product.

Now consider the case $p \geqslant q \geqslant 1$. The group $O(p, q)$ has four components and therefore $2^3 = 8$ inequivalent double coverings, which we shall call

$$\rho^{a,b,c} : Pin^{a,b,c}(p, q) \to O(p, q),$$

where $a, b, c \in \{+, -\}$ describe the squares of the elements covering reflections along the first, the last axis and their composition. Alternatively, we can use the label abc instead of c to say whether the elements covering these two reflections commute ($abc = +$) or anti-commute ($abc = -$).

Over the maximal compact subgroup $O(p) \times O(q)$ of $O(p, q)$ $\rho^{a,b,c}$ reduces to

$$\rho^{a,b,c} : (Pin^{a}(p) \times_{abc} Pin^{b}(q))/\mathbf{Z}_2 \to O(p) \times O(q)$$

$$[a, b] \to (\rho(a), \rho(b)) \tag{3.2}$$

where $\times_{abc}$ denotes the direct product with commuting (resp. anticommuting) reflections for $abc = +$ (resp. $abc = -$), $\mathbf{Z}_2 = \{(1, 1), (-1, -1)\}$ and $Pin^{a}(p)$, $a \in (+, -)$, are given by (3.1). Only two of these coverings (characterized by $a = -b$ and $abc = -$) can be obtained in terms of Clifford algebras of $\mathbb{R}^{p,q}$, namely $Pin^{+,-,+}(p, q)$ and $Pin^{-,+,+}(p, q)$. They are sometimes denoted by $Pin^{\pm}(p, q)$.

A better understanding of $Pin^{a,b,c}(p, q)$ may be gained by writing them as

$$Pin^{a,b,c}(p, q) = (Spin_0(p, q) \times_{abc} 8^{a,b,c})/\mathbf{Z}_2.$$

If we also write $O(p, q)$ as a semidirect product $SO_0(p, q) \odot (\mathbf{Z}_2 \times \mathbf{Z}_2)$ then $\rho^{a,b,c}$ assumes the form

$$[a, h] \to (\rho(a), \nu^{a,b,c}(h)), \tag{3.3}$$

where $\nu^{a,b,c} : 8^{a,b,c} \to \mathbf{Z}_2 \times \mathbf{Z}_2$ are the eight central extensions of $\mathbf{Z}_2 \times \mathbf{Z}_2$ by $\mathbf{Z}_2$. More specifically, $8^{a,b,c}$ is isomorphic with $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$, D_4 (dihedral group), $\mathbf{Z}_2 \times \mathbf{Z}_4$ or Q_4 (quaternionic group) when in the triple (a, b, c) there are respectively three, two, one or zero + signs ($8^{a,b,c}$ with $abc = -$ occur as vee groups [7] of Cliffords algebras $\mathbf{D}_4 = G_{2,0} = G_{1,1}$ and $Q_4 = G_{0,2}$, whereas $8^{a,b,c}$ with $abc = +$ do not arise in this way). Some more information can be obtained by determining the centre $Z^{a,b,c}(p, q)$ of $Pin^{a,b,c}(p, q)$.

Consider first the case $abc = +$:

p odd, q odd

$$Z^{a,b,c} = \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2 \quad \text{for } a = (-)^{(p-1)/2}, b = (-)^{(q-1)/2}$$
$$\mathbf{Z}_2 \times \mathbf{Z}_4 \quad \text{for } a = (-)^{(p-1)/2}, b = -(-)^{(q-1)/2}$$
$$\mathbf{Z}_4 \times \mathbf{Z}_2 \quad \text{for } a = -(-)^{(p-1)/2}, b = (-)^{(q-1)/2}$$
$$(\mathbf{Z}_4 \times \mathbf{Z}_4)/\mathbf{Z}_2 \quad \text{for } a = -(-)^{(p-1)/2}, b = -(-)^{(q-1)/2}$$

p odd, q even

$$Z^{a,b,c} = \mathbf{Z}_2 \times \mathbf{Z}_2 \quad \text{for } a = (-)^{(p-1)/2}$$
$$\mathbf{Z}_4 \quad \text{for } a = -(-)^{(p-1)/2}$$

p even, q odd

$$Z^{a,b,c} = \mathbf{Z}_2 \times \mathbf{Z}_2 \quad \text{for } b = (-)^{(q-1)/2}$$
$$\mathbf{Z}_4 \quad \text{for } b = -(-)^{(q-1)/2}$$

p even, q even

$$Z^{a,b,c} = \mathbf{Z}_2.$$

If $abc = -$, the anticommutativity reduces the number of elements in $Z^{a,b,c}$ and we obtain for

p odd, q odd or p even, q even

$$Z^{a,b,c} = \mathbf{Z}_2$$

p odd, q even

$$Z^{a,b,c} = \mathbf{Z}_2 \times \mathbf{Z}_2 \quad \text{for } a = (-)^{(p+q-1)/2}$$
$$\mathbf{Z}_4 \quad \text{for } a = -(-)^{(p+q-1)/2}$$

p even, q odd

$$Z^{a,b,c} = \mathbf{Z}_2 \times \mathbf{Z}_2 \quad \text{for } b = (-)^{(p+q-1)/2}$$
$$\mathbf{Z}_4 \quad \text{for } b = -(-)^{(p+q-1)/2}.$$

In the light of (3.3) standard theorems show the existence of finite dimensional irreducible representations of $Pin^{a,b,c}(p, q)$. Whether these are faithful is not a priori obvious. It is interesting to note that it is the group $Pin^{+,-,-}(3, 1)$ which plays a role in quantum electrodynamics (c.f. [8]).

If one is only interested in certain subgroups of the full orthogonal group $O(p, q)$ like the orthochronous group $O\uparrow(p, q)$ containing only «space-like» or «time-like» reflections, one can regard their double coverings as subgroups of $Pin^{a,b,c}(p, q)$ with two labels arbitrary. The notation $Pin\uparrow^{a}(p, q)$ will be used in Chapter 4 for double coverings of $O\uparrow(p, q)$.

4. EXAMPLES

We start with oriented closed two-dimensional manifolds. They are topologically classified by their genus g, i.e. the number of handles. We have already described the sphere S^2 $(g = 0)$.

Example 1

The torus $T = \mathbb{C}/\Gamma$ $(g = 1)$ is the quotient of the complex plane by a lattice Γ generated by two translations γ_1 and γ_2. Starting with the trivial spin structure on $\mathbb{C}$ it is easy to find the $4 = |\,\mathrm{Hom}(\Gamma, \mathbf{Z}_2)\,|$ structures on T since (2.1) reads

$$1 \to \mathbf{Z}_2 \to \Gamma \times \mathbf{Z}_2 \to \Gamma \to 1,$$

which certainly splits.

Example 2

A (Riemann) surface Σ_g with genus $g \geqslant 2$ is a quotient of the (contractible) upper half plane $U = (z = x + iy \in \mathbb{C} : y > 0)$ by a *Fuchsian group* Γ with $2g$ generators γ_j, $1 \leqslant j \leqslant 2g$, satisfying the relation

$$\prod_{j\ odd}^{2g-1} (\gamma_j \gamma_{j+1} \gamma_j^{-1} \gamma_{j+1}^{-1}) = 1. \tag{4.1}$$

If we let $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\mathbf{Z}_2$ act on U by fractional linear transformations

$$z \to (az + b)(cz + d)^{-1}, \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}) \tag{4.2}$$

which preseve the Poincaré metric $y^{-2} \mathrm{d}x\, \mathrm{d}y$, then Γ can be identified with a discrete subgroup of $SL(2, \mathbb{R})/\mathbf{Z}_2$ which acts on U properly discontinuously and $|\,a + d\,| > 2$ ensures that there are no fixed points. Trivializing the bundle of orthonormal frames by a global frame $e(x, y) = (y\partial/\partial x, y\partial/\partial y)$ the derivative

$\bar{\gamma}$ of γ is given by $\bar{\gamma} : (z, e) \to (\gamma(z), eR(z))$, where $R(z)$ is a rotation by the angle $2\varphi(z)$ determined by

$$exp(2i\varphi(z)) = (cz + d)(c\bar{z} + d)^{-1}.$$

The two automorphisms of $\widetilde{F} = U \times Spin(2)$ covering $\bar{\gamma}$ are given by

$$\pm \widetilde{\gamma} : (z, e) \to (\gamma(z), \pm \tilde{e}\widetilde{R}(z)),$$

where this time $\widetilde{R}(z)$ is given by $exp(i\varphi(z)\sigma_3)$. The group of such automorphisms is a nontrivial double covering of $SL(2, \mathbf{R})/\mathbf{Z}_2$ (the nontriviality can be seen by considering a loop in $SL(2, \mathbf{R})/\mathbf{Z}_2 = SO_0(2, 1)$ connecting the rotation by 2π to the identity). Hence it must be isomorphic to

$$SL(2, \mathbf{R}) \cong Spin_0(2, 1) \to SL(2, \mathbf{R})/\mathbf{Z}_2 \cong SO_0(2, 1).$$

We obtain upon restriction the exact sequence

$$\mathbf{Z}_2 \to \widetilde{\Gamma} \to \Gamma$$

which splits since the relation (4.1) can be always satisfied on the level of $SL(2, \mathbf{R})$ matrices. All the $2^{2g} = |\operatorname{Hom}(\Gamma, \mathbf{Z}_2)|$ spin structures on U/Γ are defined according to the method of chapter 2.

We pass now to closed nonorientable surfaces. Topologically they are also classified by their genus $g \geqslant 1$ which is, this time, a number of components $\mathbf{R}P_2$.

Example 3

The $g = 1$ surface $\mathbf{R}P_2$ has been already dealt with in [9]; we shall rephrase this in our terminology in light of the general construction given in chapter 2. Since $\mathbf{R}P_2 = S^2/\mathbf{Z}_2$, where $\mathbf{Z}_2$ is generated by the total inversion $J : x \to -x$ in $\mathbf{R}^3$, we start with the unique pin structure on $S^2 = 0(3)/0(2)$ (with the structure group $Pin^{\pm}(2)$), namely

$$Pin^{\pm}(3) \xrightarrow{\eta_{\pm}} 0(3) \to 0(3)/0(2) = S^2,$$

where $\eta_{\pm} = \rho_{\pm}$. The derivative $\bar{J} = TJ$ is simply given by $J : A \mapsto -A, A \in 0(3)$. It is covered by $\pm \widetilde{J} : \widetilde{A} \to \omega\widetilde{A}, \widetilde{A} \in Pin^{\pm}(3)$, where ω is the canonical volume element of the Clifford algebra underlying $Pin^{\pm}(3)$. We see that splitting of

$$(1, -1) \to (1, -1, \widetilde{J}, -\widetilde{J}) \to (1, J)$$

exists iff $\omega^2 = 1$. This condition singles out the group $Pin^-(3)$ and therefore there is no $Pin^+(2)$ structure on $\mathbf{R}P_2$, but there are the following two $Pin^-(2)$ structures

$$\eta_\pm : Pin^-(3)/\{1, \pm\omega\} \to O(3)/\{1, -1\}.$$

Example 4

A nonorientable closed surface N_g of genus $g \geqslant 3$ is a quotient of U by a NEC (*non-euclidean crystallographic*) group Γ, generated by g elements γ_i with the relation

$$\prod_{i=1}^{g} \gamma_i^2 = 1. \tag{4.3}$$

They act as a composition of $J : z \to -\bar{z}$ and the action of $SL(2, \mathbb{R})$ via (4.2) (with $a_j \neq d_j$). Now, the group generated by $SL(2, \mathbf{R})/\mathbf{Z}_2$ and J is just $O\uparrow(2, 1)$ (see the end of chapter 3). We obtain its double coverings as $Pin\uparrow^+(2, 1)$ or $Pin\uparrow^-(2, 1)$ according to whether we put a (trivial) $Pin^+(2)$, or $Pin^-(2)$ structure on U. In these two cases an element $\tilde{J}$ covering J satisfies $(\tilde{J})^2 = \pm 1$ and is explicitly given by

$$\tilde{J} : (z, \tilde{e}) \to (J(z), \tilde{e}\hat{J}(z))$$

with $\hat{J} = \sigma_1$ or $i\sigma_1$, respectively. Thus a double covering $\tilde{\Gamma}_\pm$ of Γ is precisely that subgroup of $Pin\uparrow^+(2, 1)$ or $Pin\uparrow^-(2, 1)$ which covers Γ. This incidentally shows that the relation

$$\prod_{i=1}^{g} \tilde{\gamma}_i^2 = 1 \tag{4.4}$$

(required for a splitting) can be satisfied only for one of these two subgroups if g is odd since then they differ by a sign $i^{2g} = -1$. In [10] it was shown that there are $Pin^-(2)$ but no $Pin^+(2)$ structures on closed non-orientable odd genus surfaces and thus (4.4) is satisfied for $\tilde{\Gamma}_-$. For even genus we see that there are either both $Pin^\pm(2)$ structures or none. It was shown in [10] that the former is the case.

An immediate consequence of this and of the fact that $Pin^-(2)$ has no real representations is that there are no Majorana (s)pinors on closed odd genus non-orientable surfaces (this result has been also shown in [11]).

In the rest of this section we shall discuss some examples which serve a two-fold purpose. On the one hand they illustrate our method for more complicated manifolds. On the other hand they show that the conditions for the existence of «non-Clifford» pin structures generally differ from those for $Pin^\pm(n)$ and $Pin^\pm(p, q)$. The idea of the construction is the following: the product of two non-orientable manifolds admits neither Pin^+ nor Pin^- structures (the obstacle being the non-commutativity of reflections in M_1 and M_2). But this obstacle can be overcome if we equip $M_1 \times M_2$ with a pseudo-riemannian metric and use pin structures with one of the $Pin^{a,b,c}(p, q)$ groups with $abc = +$. With this in mind we proceed to

Example 5

It is known that the product $\mathbb{R}P_2 \times \mathbb{R}P_2$ does not even admit *spin*^{c} or *pin*^{c} structures [13], but we can equip $\mathbb{R}P_2 \times \mathbb{R}P_2$ with a pseudo-riemannian metric of signature $(++--)$. We may follow the same reasoning as in the case of $\mathbb{R}P_2$ alone but now simultaneously for the two inversions J_1 and J_2 which commute. By requiring that their lifts commute as well, (this excludes the $Pin^{a,b,c}(2, 2)$ groups with $abc = -1$) we arrive at the conclusion that the only admissible structure group is $Pin^{-,-,+}(2, 2)$. The four different splittings

$$\sigma_{jk} : (1, J_1) \times (1, J_2) \to (1, j\omega_2) \times (1, k\omega_2), \text{ for } j, k \in (+, -)$$

then give rise to four pin structures on $\mathbb{R}P_2 \times \mathbb{R}P_2$. The fact that only $Pin^{-,-,+}(2, 2)$ is admissible is most easily seen in terms of maximal compact subgroups and using (3.2).

Example 6

Let us consider finally the product $K_1 \times K_2$ of two Klein bottles. Since it admits metrics of arbitrary signature $(p, 4-p)$ this will allow us to see the influence of the signature on the existence of $Pin^{a,b,c}$ structures. Furthermore, since $K_1 \times K_2$ is a $\mathbf{Z}_2 \times \mathbf{Z}_2$ quotient of $T \times T$ (see below), which itself already has $2^4 = 16$ spin structures, the subtle dependence on these will also become apparent. «Twisted» scalar and spinor fields on K have also been treated in [14]. Since the Klein bottle is obtained from T by quotienting the $\mathbf{Z}_2$-action generated by

$$J : (x_1, x_2) \to (x_1 + \pi, -x_2),$$

we have $K_1 \times K_2 = T \times T / \Gamma$, where $\Gamma = \mathbf{Z}_2 \times \mathbf{Z}_2 = (1, J_1, J_2, J_1 J_2)$ with

$$J_1 : (x_1, x_2, x_3, x_4) \to (x_1 + \pi, -x_2, x_3, x_4),$$

$$J_2 : (x_1, x_2, x_3, x_4) \to (x_1, x_2, x_3 + \pi, -x_4).$$

The inequivalent $Pin^{a,b,c}$ structures on $T \times T$ can be written down as

$$\eta : T \times Pin^{a,b,c}(p, 4-p) \to T \times O(p, 4-p).$$

$$(x, A) \to (x, R(k, x)\rho^{a,b,c}(A))$$

where $R(k, x)$ is a rotation in the 3-4 plane with winding number $k^i = 0$ or 1 along x_i. Starting for example with the trivial $(k_i = 0)$ *pin*-structure one sees that the structure groups $Pin^{+,+,+}(p, 4-p)$ with $p = 2, 3$ are allowed, each leading to $4 = |\mathrm{Hom}(\mathbf{Z}_2 \times \mathbf{Z}_2, \mathbf{Z}_2)|$ *pin*-structures on $K_1 \times K_2$.

For the twisted pin-structures ($k^i \neq 0$ for some i) the situation is as follows. There are no $Pin^{a,b,c}(3, 1)$ structures for signature $(-\,+++)$ and any k, whereas in the case $(+-++)$ we have the following admissible structure groups on $K_1 \times K_2$.

$$Pin^{+,+,+}(3,1) \text{ for } k^2 = k^1 = 0$$

$$Pin^{+,-,-}(3,1) \text{ for } k^2 = 0,\ k^1 = 1$$

$$Pin^{-,+,-}(3,1) \text{ for } k^2 = 1, k^1 = 0$$

$$Pin^{-,-,+}(3,1) \text{ for } k^2 = 1, k^1 = 1,$$

(k^1 and k^3 arbitrary) giving rise to plenty of *pin* structures. The other signatures can be dealt with analogously.

5. CONCLUSIONS AND OPEN PROBLEMS

We have classified and constructed $\widetilde{G}$-structures on quotient manifolds M/Γ in terms of data on M and illustrated this method on Riemann surfaces and certain non-orientable manifolds. Furthermore we have studied the 8 inequivalent double coverings of the pseudo-orthogonal groups $O(p, q)$, $p \geqslant q \geqslant 1$, emphasizing in particular their finite group structure and their relevance for defining «pinors» on non-orientable manifolds. There remain however some open problems, which deserve attention:

a) Do there exist finite dimensional *faithful* irreducible representations of $Pin^{a,b,c}(p, q)$?. Preliminary investigations seem to indicate that this may not be the case e.g. for the group $Pin^{+++}(3, 1)$.

b) What are the topological obstructions to the existence of a $Pin^{a,b,c}$ structure on a manifold M?. Our examples (section 4) only indicate that they certainly differ from the obstructions for $pin^{\pm}$-structures.

c) Can the theorem of section 2 be extended in a meaningful way to handle orbifolds as well? This question has been discussed at the end of section 2, where the limitations of this extension have been indicated.

d) Can one suggest a gedanken experiment which could be sensitive to the labels a, b, c? One can write the analogue of Dirac operator for «non-Clifford» groups $Pin^{a,b,c}$; the square of this operator will however contain mixed derivatives which does not follow the original idea of Dirac.

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